Problem 1. (Strang 2.1.5) Recall that a vector space \( V \) is a collection \( V = \langle V, K, \cdot, + \rangle \) which satisfies the ten rules \( V1-V10 \) given in class (Strang combines some of them to get eight rules).

1. Taking \( V = \mathbb{R}^2, K = \mathbb{R} \), suppose that addition in \( \mathbb{R}^2 \) adds an extra one to each component, so that \((3,1) + (5,0) = (9,2) \) rather than \((8,1) \). With scalar multiplication unchanged, which vector space rules are broken?

2. Taking \( V = \mathbb{R}, K = \mathbb{R} \), show that the set of all positive real numbers, with \( x + y \) and \( c \cdot x \) redefined to the usual \( xy \) and \( x^c \), respectively, is a vector space. What is the ”zero” vector in this new space?

Problem 2. (Strang 2.3.1) Decide (rigorously) whether or not the following vectors are (1) linearly independent and (2) span \( \mathbb{R}^3 \).

\[
v^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad v^4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.
\]

Problem 3. (Strang 2.4.3) Find the dimension and a basis of the four fundamental subspaces for both

\[
A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Problem 4. Let \( V = \text{span}\{v^1, v^2\} \subseteq \mathbb{R}^2 \), and let \( W = \text{span}\{w^1, w^2\} \subseteq \mathbb{R}^3 \), where

\[
v^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad w^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w^2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.
\]

Construct the matrix representation \( A \) of the linear transformation which maps \( v^1 \) to \( w^1 \), and maps \( v^2 \) to \( w^2 \). If we change the basis for \( V \) to

\[
v^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix},
\]

how does the matrix \( A \) change?
• **Problem 5.** (Strang 2.6.8) The space $\mathcal{P}_3$ of cubic polynomials is a vector space, and a standard basis is $\{1,t,t^2,t^3\}$. We know that differentiation and integration are “linear” transformations to functions, so we should be able to construct the matrix representation of these operations on $\mathcal{P}_3$ in this standard basis.

1. Construct the matrix $A$ representing $d^2/dt^2$.
2. What are the null and range spaces of $A$?
3. What is the interpretation of these spaces in terms of polynomials?

• **Problem 6.** Is the following system uniquely solvable? (If so, determine the solution)?

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
1 \\
4 \\
4
\end{pmatrix}.
$$

What if the right-hand side is changed to $(1, 4, 3)^T$? If it is solvable now, determine a solution. If the solution is not unique, determine a second solution. If you didn’t approach this problem using the Fredholm alternative presented in class, then re-work the problem using the Fredholm alternative argument.

• **Problem 7.** Show that the function $d(u,v) : \mathbb{R}^n \to \mathbb{R}$, defined by

$$
d(u,v) = \|u - v\|_2^2 = \left(\sum_{i=1}^{n} |u_i - v_i|^2\right)^{1/2}
$$

(where $\| \cdot \|_2$ is the usual Euclidean 2-norm) satisfies the three properties of a metric, and therefore gives $\mathbb{R}^n$ the topological structure of a metric space.

• **Problem 8.** In class, we connected vectors and linear operators on vector spaces with linear functionals and bilinear forms (respectively) through the Riesz representation theorem and the bounded operator theorem. Using these ideas, give a completely rigorous argument to conclude that if $A \in L(H,H)$ (i.e., $A$ is a bounded linear operator on a finite-dimensional Hilbert space $H$), then the problem:

Find $u \in H$ such that $Au = F \in H$

is equivalent to the problem

Find $u \in H$ such that $a(u,v) = f(v)$, $\forall v \in H$, where $A$ is related to the bilinear form $a(\cdot,\cdot)$ through the bounded operator theorem, and $F$ is related to the linear form $f(\cdot)$ through the Riesz theorem. (For example, this allows us to replace a matrix equation with an equivalent equation involving linear and bilinear functionals when convenient.)

• **Problem 9.** In class a projection operator $P$ was defined in terms of a direct sum, and as a consequence we saw that a linear operator $P$ was a projection iff it was idempotent, i.e., $P^2 = P$. Prove that a linear operator $P$ is a projection iff the operator $I - P$ is a projection, where $I$ is the identity operator.