AMa105b

Homework #4 (Hyperbolic Equations)

Handed out: Friday, May 31, 1996
Due in my office: Friday (at noon), June 7, 1996

- **Problem 1.** Consider the following two-dimensional hyperbolic equation:

  \[ u_t + u_x + u_y = 0, \]

  with appropriate boundary and initial conditions for well-posedness.

  1. Write down an explicit numerical scheme that does upwinding in both the \( x \)- and \( y \)-directions.

  2. Put the scheme into a matrix formulation; i.e., the new vector of solution values is generated by the old vector by a matrix-vector multiplication, as in the parabolic case in Homework #3.

  3. Formulate a sufficient condition for stability of the scheme, using matrix and vector norms rather than individual Fourier modes. (I.e., think about what it will take to keep the norm of the vector of solution values from blowing up.)

- **Problem 2.** Consider the following system of partial differential equations:

  \[ \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad \text{in} \ \mathbb{R} \times (0,T], \]

  where \( u(x,t) \) and \( f(x,t) \) are vector functions:

  \[ u(x,t) = [u_1(x,t), u_2(x,t), \ldots, u_n(x,t)]^T, \quad f(x,t) = [f_1(x,t), f_2(x,t), \ldots, f_n(x,t)]^T. \]

  Such a set of equations is called a *system of conservation laws*.

  1. Show that such a system can be written in the form

     \[ u_t + A(u)u_x = 0, \]

     and write out the form of the matrix function \( A(u) \).

  2. Assume now that for any fixed \( u \), the matrix \( A(u) \) has positive real eigenvalues, and also has a full set of linearly independent eigenvectors. We can construct an \( n \times n \) matrix \( S \) by taking as its columns the eigenvectors of \( A(u) \), and we can write \( \Lambda \) as a diagonal matrix with the eigenvalues of \( A(u) \) on the diagonal. Derive now the characteristic normal form

     \[ Su_t + \Lambda Su_x = 0. \]

  3. Assume that \( A(u) = A \) is a constant positive definite matrix, and that you employ the simple upwind differencing approach to this problem. What does the sufficient condition for stability that you derived in Problem #1 look like in this setting?
• **Problem 3.**

Consider the Wave Equation

\[ u_{tt} = c^2 u_{xx} \quad \text{on} \quad 0 \leq x \leq L, \]

with

\[ u(0, t) = u(L, t) = 0, \quad u_t(0, t) = u_t(L, t) = 0. \]

Show that the following function (which is proportional to the total energy) is conserved:

\[ E = \int_0^L \left( (cu_x)^2 + (u_t)^2 \right) \, dx. \]

• **Problem 4.** (EXTRA CREDIT)

Consider the wave equation from Problem #3, written as a first order system:

\[ v = u_t, \quad v_t = u_{xx}. \]

The equation is supplied with the side conditions from Problem #3, and assume also the following initial condition:

\[ u(x, 0) = \frac{1}{2} x (L - x), \quad u_t(x, 0) = 0. \]

Are the boundary and initial conditions consistent? If so, what does the solution to this problem like look as time evolves? Hint: The weak formulation of the problem (derived in the same way as in the elliptic case from class) may be helpful here in analyzing the situation.